Semianalytic Theory of Long-Term Behavior of Earth and Lunar Orbiters

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A semianalytical method capable of analyzing both lunar and Earth orbiters is presented. Primary attention is focused on predicting the evolution of the orbit as affected by third body perturbations together with those of the rotating primary. The singly averaged (literal) equations of motion are expanded by machine to high order in the parallax factor and the mean motion ratio. The equations are numerically integrated to yield the orbital evolution for a wide range of initial conditions. In addition, a purely analytical method is introduced to yield the orbital lifetimes for a special class of orbits.

Introduction

THE design of any mission to place a satellite in orbit about the Earth or moon is a complex and time consuming investigation of the types of orbits that would meet stated objectives. Because of the complex dynamics field in the Earth-moon system, the mission analysis phase of the study must and will of necessity, include a time history of many orbits in order to gain the maximum scientific data from the final orbit chosen. Since a variety of initial conditions will be used, it is essential that the model chosen to produce the time history be not only accurate but fast. Any good n-body precision integration program is capable of meeting the first of these criteria but certainly not the second.

To speed up the computations involved, the standard approximation used has been to doubly average the disturbing function due to the presence of a third body. First the disturbing function is averaged over one orbit of the satellite and then over one revolution of the central body about the disturbing body. This process eliminates all short and medium-period terms leaving only the long-period perturbations for consideration. However, experience has shown this model has limited use for the Earth moon-system. Also for a planet such as Mars where coupling between oblateness and the third body can be strong, the doubly averaged system sometimes fails. A good discussion of this model can be found in Ref. 1.

The singly averaged equations of motion have been shown in Ref. 2 to be highly accurate for orbiters of Mars. Here only the short-period terms are averaged out and the equations of motion retain the medium and long-period terms. Therefore the model is also valid for both near and far orbiters of the Earth and moon. However, as shown in that reference, the expansion of the third body disturbing function is essentially in terms of the parallax factor (a/r') only. For a 100,000 km high Earth orbiter this ratio is about 0.25 for the moon as the disturbing body. The expansion in Ref. 2 was truncated to retain only terms of second order in the parallax factor but for high Earth or lunar orbiters this is not sufficient. The expansion must be to at least fourth order and for high orbiters (100,000 km) should be carried even further. Further discussions and uses of the singly averaged equations may be found in Refs. 3 and 4.

When the third body terms are averaged, the assumption is sometimes made that the disturbing body does not move significantly over one orbit of the satellite, however, for high orbits this assumption is clearly violated in the case of the Earth-moon system. Thus in carrying out the expansion, a time rate of change for all terms containing the third body position must be included. This yields a further expansion of the disturbing function in terms of the mean motion ratio n'/n; the ratio of the mean motion of the disturbing body to that of the satellite.

To carry out the expansions in terms of the parallax factor and the mean motion ratio and then to average the equations of motion over one orbit requires an excessive amount of algebra for the higher order terms. To aid in the algebraic computations, a general algebraic manipulation routine was developed and was used to compute the average (literal) equations of motion to eighth order in the parallax factor and to second order in the mean motion ratio with corresponding cross terms up to and including fifth order in parallax.

In addition to the third body effects, the gravity harmonics of the rotating primary must be considered. For the moon, these equations may be averaged over the orbital period since the moon rotates slowly. However, for the more rapidly rotating Earth, this analysis is invalid as the orbital mean motion may be nearly commensurate with the rotation of the primary. To avoid this problem, the equations of motion are numerically averaged from one-half orbit behind to one-half orbit ahead of the present position of the satellite. These averaged rates are then used in the total variations of the elements. At present a full 7×7 and 4×4 field is used for the Earth and moon, respectively. When the tesseral harmonics are not required, only terms containing J_2, J_2^2, J_3 and J_4 are used and the variational equations are calculated explicitly.

In addition to this semianalytic approach, a purly analytical model is also developed. This model concentrates on the long-period terms and applies to both lunar and high Earth orbiters. To derive the long-period equations of motion, the medium-period terms must be removed. This can be done provided the Earth and moon move much faster on their respective paths than the line of apsides of the perturbed orbit under question. This condition is satisfied for all lunar orbiters provided their height does not exceed approximately four lunar radii. Thus these medium-period effects involving the Earth angle can be removed by a von Zeipel transformation.

For Earth orbiters, the analysis is more complicated. The line of apsides rotates with an angular velocity of about 8° per day for a low orbiter, whereas the sun and moon move along their respective paths with angular rates of 1° and 13° per day, respectively. Thus these medium-period terms cannot be removed for the low orbiters; however, the analysis is valid for the higher orbits and useful results can be obtained.

In the absence of oblateness, the solution to the equations of motion can be expressed in closed form using elliptic integrals. However, in certain special cases involving initially near circular orbits, the solution involves only the elementary

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functions. These special cases are extended to include oblateness by introducing a series expansion. The solutions are accurate and yield results applicable to initially near circular orbits. These solutions yield the long-period time history of eccentricity and pericenter position and give the orbital lifetime for unstable orbits. This procedure works well for high lunar orbits and can be extended to high earth orbiters.

Machine Automated Algebra

The use of machine automated algebra in celestial mechanics has become increasingly popular in recent years. Due to the high probability of error which is introduced when hand methods are used it was decided to develop the equations of motion to high order by computer. To this end it was decided to construct an algebraic manipulation program compatible with the CDC 3800 presently in use at the Naval Research Lab. This program will manipulate the otherwise involved literal Poisson series occurring in classical perturbation theory. The computerized operations include the simplification, ordering, negation, addition, subtraction, multiplication, differentiation and integration of the trigonometric series occurring in the theory. Other more specialized routines include a binomal and Taylor series expansion. The program is written in Fortran and can be used on any machine possessing a Fortran compiler with little or no modifications.

The equations of motion (to be described later) were developed entirely by computer. These literal equations were automatically card punched in Fortran compatible form and inserted directly into the variation of parameters program with no human interaction. Literally thousands of terms were involved in these expansions with a savings in time of many months and possibly years. The entire expansion required approximately 2 min, of central processing time on the CDC 3800. The method has the added advantage in that trivial algebraic and keypunching errors are eliminated as possible errors in the analysis. Remaining errors can be traced to those of concept and the programming of the literal expansions. A detailed description of the algebraic manipulation program may be found in Ref. 5.

Variation of Parameters

The equations for the variation of the keplerian elements are well known and are developed in any good textbook on celestial mechanics. Therefore the equations will be listed here without derivation in the two forms that were used in the computer program.

Lagrange's Planetary Equations

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial F}{\partial M} \\
\frac{de}{dt} &= \frac{(1-e^2)^{1/2}}{ena^2}\left[(1-e^2)^{1/2}\frac{\partial F}{\partial M} - \frac{\partial F}{\partial \omega}\right] \\
\frac{d\Omega}{dt} &= \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial F}{\partial i} \\
\frac{di}{dt} &= -\frac{\csc i}{na^2(1-e^2)^{1/2}}\left[\frac{\partial F}{\partial \Omega} - \cos i\,\frac{\partial F}{\partial \omega}\right] \\
\frac{d\omega}{dt} &= -\frac{(1-e^2)^{1/2}}{ena^2}\frac{\partial F}{\partial e} - \frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial F}{\partial i} \\
\frac{dM}{dt} &= n - \frac{2}{na}\frac{\partial F}{\partial a} - \frac{(1-e^2)}{nea^2}\frac{\partial F}{\partial e}
\end{aligned}
\tag{1}
$$

Gauss' Form of the Equations

$$
\begin{aligned}
\frac{da}{dt} &= \frac{2}{n(1-e^2)^{1/2}}\left[Re\sin f + \frac{a(1-e^2)}{r}S\right] \\
\frac{de}{dt} &= \frac{(1-e^2)^{1/2}\sin f}{na}R + \frac{(1-e^2)^{1/2}}{ena^2}\left[\frac{a^2(1-e^2)-r^2}{r}\right]S \\
\frac{d\Omega}{dt} &= \frac{r\sin(\omega+f)}{na^2(1-e^2)^{1/2}\sin i}W \\
\frac{di}{dt} &= \frac{r\cos(\omega+f)}{na^2(1-e^2)^{1/2}}W \\
\frac{d\omega}{dt} &= -\frac{(1-e^2)^{1/2}}{ane}\cos fR + \frac{(1-e^2)^{1/2}\sin f}{ane}\times \\
&\quad \left[1+\frac{r}{a(1-e^2)}\right]S - \frac{r\sin(\omega+f)\cot i}{a^2n(1-e^2)^{1/2}}W \\
\frac{dM}{dt} &= n + \left[\frac{(1-e^2)\cos f}{ane} - \frac{2r}{na^2}\right]R - \\
&\quad \frac{(1-e^2)\sin f}{ane}\left[1+\frac{r}{a(1-e^2)}\right]S
\end{aligned}
\tag{2}
$$

Where the disturbing force $\vec{\Delta}$ is defined as

$$\vec{\Delta} = R\vec{U}_r + S\vec{U}_\theta + W\vec{U}_A \tag{3}$$

and $\vec{\Delta}$ is decomposed into components in the radial, (R), transverse, (S), orbit plane normal, (W), directions where

$$\vec{U}_A = \vec{U}_r \times \vec{U}_\theta$$

The Third Body Disturbing Function

The acceleration experienced by a satellite under the influence of a point mass third body is

$$\ddot{\vec{r}}_I = \ddot{\vec{r}} - \ddot{\vec{r}}' = \nabla(\mu'/|\vec{r}' - \vec{r}|) \tag{4}$$

where μ' is the gravitational coefficient of the third body and $\ddot{\vec{r}}_I$ denotes the acceleration vector in inertial space. $\vec{r}'$ and $\vec{r}$ are the position vectors to the third body and the satellite respectively (see Fig. 1.)

Equation (4) may be expressed as

$$\ddot{\vec{r}} = -\mu'\nabla\left(\frac{1}{|\vec{r}'-\vec{r}|} - \frac{\vec{r}'\cdot\vec{r}}{r'^3}\right) \tag{5}$$

where

$$\ddot{\vec{r}}' = \frac{\mu'}{r'^3}\vec{r}' = \mu'\nabla\left(\frac{\vec{r}'\cdot\vec{r}}{r'^3}\right)$$

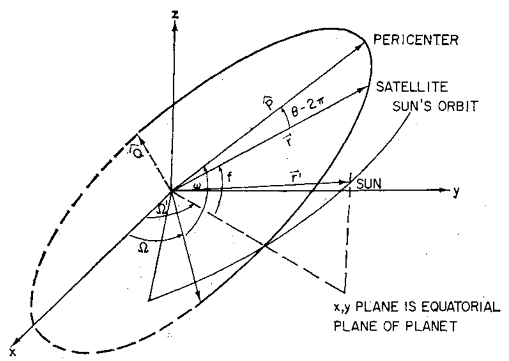

Fig. 1 Planet-centered geometry.

Equation (5) can be written in the following form

$$\vec{\ddot{r}} = \nabla F(\vec{r}, \vec{r}') \tag{6}$$

where

$$F(r, r') = \frac{\mu'}{r'}\left[\frac{1}{\left(1 - 2\frac{r}{r'}\cos S + \left(\frac{r}{r'}\right)^2\right)^{1/2}} - \left(\frac{r}{r'}\right)\cos S\right] \tag{7}$$

and

$$\cos S = \vec{r}\cdot\vec{r}'/rr'$$

Now introduce the eccentric anomaly E directly into Eq. (7), i.e., let $(r/r')\cos S = \delta A(\cos E - e) + \delta B(1-e^2)^{1/2}\sin E$, $(r/r')^2 = \delta^2(1 - e\cos E)^2$, $\delta = a/r' =$ parallax factor, $A = \hat{P}\cdot\hat{r}'$ and $B = \hat{Q}\cdot\hat{r}'$ where $\hat{P}$ and $\hat{Q}$ are shown in Fig. 1 and $\hat{r}'$ is the unit vector to the third body.

$F(\vec{r}', \vec{r})$ was expanded directly to order 8 in δ by using the algebraic manipulation program previously mentioned. Due to the direct nature of the expansion, the explicit expressions for the Legrendre polynomials were not required. The coefficients of each factor $\delta^n(2 \leqslant n \leqslant 8)$ were automatically collected and the disturbing function is then obtained in the form

$$F(r, r') = \frac{\mu'}{r'}\sum_{n=2}^{8}\delta^n F_n(A, B, e, E) \tag{8}$$

where each $F_n(A, B, e, E)$ is of the form

$$F_n(A, B, e, E) = \sum_{m=0}^{n} C_{jklm}A^jB^ke^l\cos mE + \sum_{m=1}^{n} S_{jklm}A^jB^ke^l\sin mE \tag{9}$$

where both C_{jklm} and S_{jklm} are obtained as rational integer coefficients. The equations of motion as determined by the Lagrangian planetary equations are

$$\dot{x}_i = f_i(A, B, \gamma, e, E)(1 \leq i \leq 6) \tag{10}$$

where

$$x_i = \text{Keplerian state vector}$$

$$\gamma = \begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}; \qquad \begin{aligned}\gamma_1 &= \hat{R}\cdot\hat{r}'\\ \gamma_2 &= \vec{P}_1\cdot\hat{r}'\\ \gamma_3 &= \vec{Q}_1\cdot\hat{r}'\end{aligned}$$

where

$$\vec{P}_1 = \begin{pmatrix}-P(2)\\ P(1)\\ 0\end{pmatrix}; \qquad \vec{Q}_1\begin{pmatrix}-Q(2)\\ Q(1)\\ 0\end{pmatrix}; \quad \text{and} \quad \hat{R} = \hat{P}\times\hat{Q}$$

Thus far the third body analysis is exact except for Eq. (8) where the expansion was terminated at order 8 in the parallax factor. The next step is to average the equations of motion over one orbital period, i.e.,

$$\bar{\dot{x}}_i = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_i\,dM = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_i(1 - e\cos E)\,dE \tag{11}$$

In the first order variation of parameters analysis it may be assumed that the orbital elements $(a, e, i, \Omega, \omega)$ are constant as the equations of motion are averaged. However, the third body position cannot be held constant.

Therefore, A, B, and γ are expanded about their nominal values, i. e.,

$$A = A_0 + \frac{\partial A}{\partial\theta'}\left(\frac{n'}{n}\right)(M - M_0) + \frac{1}{2}\frac{\partial^2 A}{\partial\theta'^2}\left(\frac{n'}{n}\right)^2(M - M_0)^2 + \ldots \tag{12}$$

with similar expressions for B and γ. θ' is the third body central angle and

$$\frac{\partial A}{\partial\theta'} = \frac{1}{n'}\hat{P}\cdot\frac{d\hat{r}'}{dt}$$

and $d\hat{r}'/dt$ is given along with $\vec{r}'$ by an analytical ephemeris. Equation (10) becomes

$$\dot{x}_i = f_i(A_0, B_0, \gamma_0, e, E) + \frac{\partial f_i}{\partial A}\frac{\partial A}{\partial\theta'}(n'/n)(M - M_0) + \frac{\partial f_i}{\partial B}\frac{\partial B}{\partial\theta'}(n'/n)(M - M_0) + \frac{\partial f_i}{\partial\gamma}\frac{\partial\gamma}{\partial\theta'}(n'/n)(M - M_0) + \text{higher order terms} \tag{13}$$

Substitution of this into Eq. (11) then yields the averaged rates of change for the elements.

Since the disturbing function is time dependent, the time rate of change of the semimajor axis has a nonzero average. Although no secular change results, small but not insignificant twice monthly and twice yearly fluctuations in the semimajor axis do appear.

The final form of the equations of motion are obtained in the parallax and mean motion ratio as follows:

$$\bar{\dot{x}}_i = \frac{\mu'}{r'}\left[\sum_{k=2}^{8} f_{k0}(a/r')^k + \sum_{k=2}^{5} f_{k1}(a/r')^k(n'/n) + f_{22}(a/r')^2(n'/n)^2\right] \tag{14}$$

where f_{k0}, k_{k1} and f_{22} are determined from the averaged equations of motion and are too lengthy to be listed here but may be found in Ref. 6.

For a very high Earth or lunar orbiter it is not enough to assume that the orbital elements $(a, e, i, \Omega, \omega)$ remain constant during the averaging process as was done here. For these high orbits it is necessary to include the coupling between the short-period fluctuations in the elements with the short-periodic part of the disturbing function. This is especially important for high lunar orbiters. These additional expansions are being carried out but have not yet been implemented in the variation of parameters program.

Gravitational Field Analysis

For a satellite of the moon, the disturbing function could be averaged analytically, however, for most other planets such is not the case. Representing the gravitational field of the body by the standard expansion in spherical harmonics, we have

$$U = \frac{\mu}{r} + F = \frac{\mu}{r}\left\{1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{R^n}{r^n}P_n{}^m(\sin\phi)\times[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda]\right\} \tag{15}$$

Then in the Lagrangian equations we would normally substitute

$$\bar{F} = \frac{1}{\tau}\int_0^{\tau} F dt$$

for the disturbing function where, more explicitly

$$\bar{F} = \frac{\mu(1-e^2)^{3/2}}{P^{n+1}}\frac{R^n}{2\pi}\int_0^{2\pi}(1 + e\cos f)^{n+1}\frac{P_n{}^m(\sin\phi)}{\cos^m\phi}\times[\cos^m\phi e^{jm(\lambda-\theta)}]df \tag{16}$$

where $\theta = \dot{\theta}t$ and $\dot{\theta}$ is the rotation rate of the primary.

The integral in the above equation is evaluated with all quantities held constant except f. For a satellite of the moon, the change in θ over one orbit is negligible to the first approximation, but for other planets, this change can be significant. It is for this reason, that the disturbing accelerations caused by a nonspherical central body must be treated differently. Here the equations of motion are averaged numerically and these averaged equations are then integrated to obtain the variation in the orbital elements. This treatment was suggested by Lorell[4] and Uphoff.[7] Uphoff defines the averaged Lagrangian equations of motion in the following manner: let $\dot{x}_i$ be the standard variational equation of any element. The averaged Lagrangian equation is then

$$\bar{\dot{x}}_i = \frac{1}{T}\int_0^{\tau} \dot{x}_i(\tau)dt = \frac{n}{2\pi}\int_0^{2\pi} \dot{x}_i(f)\frac{dt}{df}df \tag{17}$$

but

$$df/dt = (\mu P)^{1/2}/r^2$$

therefore

$$\bar{\dot{x}}_i = \frac{nP^2}{2\pi(\mu P)^{1/2}}\int_0^{2\pi} \frac{\dot{x}_i(f)df}{(1+e\cos f)^2} \tag{18}$$

Substituting the Gaussian form of the variational equations into Eq. (18), the following is obtained:

$$\begin{aligned}
\bar{\dot{a}} &= \frac{a^2(1-e^2)}{\pi(\mu a)^{1/2}}\int_{f_1}^{f_2} \frac{e\sin f\,R(f) + (1+e\cos f)S(f)}{(1+e\cos f)^2}df \\
\bar{\dot{e}} &= \frac{nP^2}{2\pi\mu}\int_{f_1}^{f_2}\left[\frac{\sin f}{(1+e\cos f)^2}R(f) + \right. \\
&\qquad \left.\frac{2\cos f + e(1+\cos^2 f)}{(1+e\cos f)^3}S(f)\right]df \\
\bar{\dot{i}} &= \frac{nP^2}{2\pi\mu}\int_{f_1}^{f_2}\frac{\cos(\omega+f)W(f)}{(1+e\cos f)^3}df \\
\bar{\dot{\omega}} &= \frac{nP^2}{2\pi\mu}\int_{f_1}^{f_2}\left[\frac{(2+e\cos f)\sin f}{e(1+e\cos f)^3}S(f) - \right. \\
&\qquad \left.\frac{\cos f}{e(1+e\cos f)^2}R(f) - \frac{\sin(\omega+f)\cos i}{(1+e\cos f)^3\sin i}W(f)\right]df \\
\bar{\dot{\Omega}} &= \frac{nP^2}{2\pi\mu}\int_{f_1}^{f_2}\frac{\sin(\omega+f)}{(1+e\cos f)^3\sin i}W(f)df \\
\bar{\dot{M}} &= n + \frac{nP^2(1-e^2)^{1/2}}{2\pi\mu}\int_{f_1}^{f_2}\left\{\left[\frac{\cos f}{e} - \frac{2}{(1+e\cos f)}\right] \times \right. \\
&\qquad \left.\frac{R(f)}{(1+e\cos f)^2} - \frac{(2+e\cos f)\sin f}{e(1+e\cos f)^3}S(f)\right\}df
\end{aligned} \tag{19}$$

The evaluation of the definite integrals in these equations is done using Gaussian quadratures with 24 points.

The computation of the disturbing accelerations is accomplished using a technique devised by DeWitt in Ref. 8. This method was programmed to include any order of the zonals and tesserals desired. At present a full 7×7 and 4×4 field is used for the Earth and moon respectively. The accelerations are evaluated in cartesian coordinates and transformed to components in the radial (R), transverse (S), and orbit plane normal (W), directions by means of the following rotation transformations

$$T_{R,S,W} = T_z(\omega + f)\,T_x(i)\,T_z(\Omega)$$

The transformed accelerations are then used directly in Eqs. (19).

For central bodies other than the Earth or moon, only terms containing J_2, $J_2{}^2$, J_3 and J_4 are used and the variational equations are calculated explicitly without going through the above averaging and quadratures. The variational equations will simply be listed here. The equations in J_2 and $J_2{}^2$ are derived in detail in Ref. 9 with additional terms from Ref. 10 and those in J_3 and J_4 were taken from Ref. 11. This option is also used for high-speed analysis of Earth and lunar orbiters where a full gravity field is not required.

Equations in J_2, $J_2{}^2$

$$\begin{aligned}
\left(\frac{da}{dt}\right)_{J_2} &\equiv 0 \\
\left(\frac{de}{dt}\right)_{J_2} &= -\frac{45nJ_2{}^2R_e{}^4}{32P^4}e(1-e^2)\sin^2 i\sin 2\omega\left(\frac{14}{15} - \sin^2 i\right) \\
\left(\frac{d\Omega}{dt}\right)_{J_2} &= -\frac{3nJ_2R_e{}^2\cos i}{2P^2} - \frac{9nJ_2R_e{}^4\cos i}{4P^4}\left[\frac{3}{2} - \frac{5}{3}\sin^2 i + \right. \\
&\qquad e^2\left(\frac{1}{6} + \frac{5}{24}\sin^2 i\right) + \frac{e^2\cos 2\omega}{4}\left(\frac{7}{3} - 5\sin^2 i\right) + \\
&\qquad \left.(1-e^2)^{1/2}\left(1 - \frac{3}{2}\sin^2 i\right)\right] \\
\left(\frac{di}{dt}\right)_{J_2} &= \frac{45}{64}\frac{J_2{}^2R_e{}^4}{P^4}ne^2\sin 2i\sin 2\omega\left(\frac{14}{15} - \sin^2 i\right) \\
\left(\frac{d\omega}{dt}\right)_{J_2} &= \frac{3nJ_2R_e{}^2}{2P^2}\left(2 - \frac{5}{2}\sin^2 i\right) + \\
&\qquad \frac{3nJ_2{}^2R_e{}^4}{16P^4}\left\{48 - 103\sin^2 i + \frac{215}{4}\sin^4 i + \right. \\
&\qquad e^2\left(7 - \frac{9}{2}\sin^2 i - \frac{45}{8}\sin^4 i\right) - \\
&\qquad \cos 2\omega\left[\left(7 - \frac{15}{2}\sin^2 i\right)\sin^2 i - \right. \\
&\qquad \left. e^2\left(7 - \frac{79}{2}\sin^2 i + \frac{135}{4}\sin^4 i\right)\right] + \\
&\qquad \left.(1-e^2)^{1/2}(24 - 66\sin^2 i + 45\sin^4 i)\right\}
\end{aligned} \tag{20}$$

where R_e is the equatorial radius of the planet, $P = a(1-e^2)$ is the semilatus rectum, and the subscript J_2 means oblateness and J_2 terms only.

Equations in J_3

$$\begin{aligned}
\left(\frac{dP}{dt}\right)_{J_3} &= 2P\tan i\left(\frac{di}{dt}\right)_{J_3} \\
\left(\frac{de}{dt}\right)_{J_3} &= -\frac{3}{8}\frac{nR_e{}^3J_3}{P^3}(1-e^2)\cos\omega\sin i \times \\
&\qquad (5\cos^2 i - 1) \\
\left(\frac{d\Omega}{dt}\right)_{J_3} &= \frac{3nR_e{}^3J_3}{3P^3}e\sin\omega\cot i(15\cos^2 i - 11) \\
\left(\frac{di}{dt}\right)_{J_3} &= \frac{3nR_e{}^3J_3}{8P^3}e\cos\omega\cos i(5\cos^2 i - 1) \\
\left(\frac{d\omega}{dt}\right)_{J_3} &= \frac{3nR_e{}^3J_3}{8P^3}\frac{(1+4e^2)}{e}\sin\omega\sin i \times \\
&\qquad (5\cos^2 i - 1) - \left(\frac{d\Omega}{dt}\right)_{J_3}\cos i
\end{aligned} \tag{21}$$

where

$$\left(\frac{da}{dt}\right)_{J_3} = \frac{(dP/dt)_{J_3} + 2ae\,(de/dt)_{J_3}}{(1-e^2)} \tag{22}$$

Equations in J_4

$$\left(\frac{dP}{dt}\right)_{J_4} = 2P \tan i \left(\frac{di}{dt}\right)_{J_4}$$

$$\left(\frac{de}{dt}\right)_{J_4} = -\frac{15nR_e{}^4 J_4}{32P^4} e(1-e^2)\sin 2\omega \sin^2 i \times (7\cos^2 i - 1)$$

$$\left(\frac{d\Omega}{dt}\right)_{J_4} = \frac{15nR_e{}^4 J_4}{32P^4}\cos i\{2(7\cos^2 i - 3) + e^2[7\cos^2 i - 1 + 4\sin^2\omega(7\cos^2 i - 4)]\} \quad (23)$$

$$\left(\frac{di}{dt}\right)_{J_4} = \frac{15nR_e{}^4 J_4}{64P^4} e^2 \sin 2\omega \sin 2i(7\cos^2 i - 1)$$

$$\left(\frac{d\omega}{dt}\right)_{J_4} = -\frac{15nR_e{}^4 J_4}{16P^4}\left\{8 - 28\sin^2 i + 21\sin^4 i - \sin^2\omega \sin^2 i(7\cos^2 i - 1) + e^2\left[6 - 14\sin^2 i + \frac{63}{8}\sin^4 i + \sin^2\omega\left(6 - 35\sin^2 i + \frac{63}{2}\sin^4 i\right)\right]\right\}$$

where a definition identical to Eq. (22) holds with the proper change in the subscripts.

Drag

For the variations due to the presence of an atmosphere, a model has been assumed in which there are no lift forces present, the drag force acts as a negative tangential component due to a nonrotating atmosphere. This results in variations only in the semimajor axis and eccentricity. The variational equations are then averaged over a single orbit using Gaussian quadratures. The density is taken from several models and is calculated as a function of altitude. These models can be found in Refs. 12 and 13 for Mars and Venus, respectively. For the Earth, no atmosphere was used in the present program although it would be very easy to incorporate a model similar to the type used for Mars and Venus.

The detailed derivation may be found in Ref. 2 with only the results being listed here.

$$\bar{a} = -\frac{C_D A a^2(1-e^2)^{3/2}}{2\pi m\mu}\int_{-\pi}^{\pi}\frac{\rho V^3}{(1+e\cos f)^2}df \quad (24)$$

$$\bar{e} = -\frac{C_D A(1-e^2)^{3/2}}{2\pi m}\int_{-\pi}^{\pi}\frac{\rho V(e+\cos f)}{(1+e\cos f)^2}df \quad (25)$$

where C_D is the aerodynamic drag coefficient, A is the cross-sectional area and ρ is the density.

It is to be stressed that only a nonrotating atmosphere has been considered; otherwise di/dt and $d\Omega/dt$ would be nonzero. The complexities of assuming an exponential density profile have also been bypassed by averaging the effects over one revolution of the satellite.

Sample Cases

The method described above has been programmed for a CDC 3800 computer in double precision mode under the program name of POPLAR: Planetary Orbiter Prediction and Lifetime Analysis Routine. Figure 2 shows a comparison between POPLAR and an Encke n-body numerical integration program for a lunar orbiter. Most of the slight differences noticed here are due to the fact that mean elements were not used and to differences in constants between the two programs. However, it is important to note that these differences do not appear to be growing with time. The numerical integration

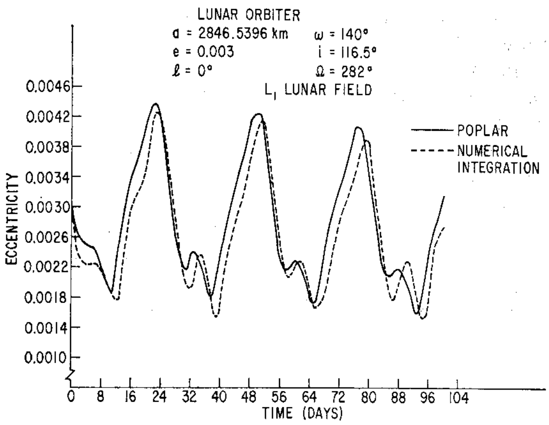

Fig. 2 Eccentricity vs time-POPLAR and numerical integration comparison.

program took 4.6 min of 360/95 time which translates to about 115 min of 3800 time. POPLAR took 3.65 min for the same case—a factor of over thirty to one.

Figure 3 is a plot of eccentricity vs argument of pericenter for a lunar orbiter. Superimposed on this graph are contours of constant lifetime. The value of such a plot is that it allows initial conditions to be selected that will yield any given lifetime. For lunar orbiters, the inclination does not vary by more than a degree or two and, therefore, these curves may be used as a very good first approximation to a nominal orbit. The data shown in Fig. 3 is composed of 13 different cases and took a total of about 15 min of 3800 CPU time.

Figure 4 is a similar curve for an Earth orbiter of 75°

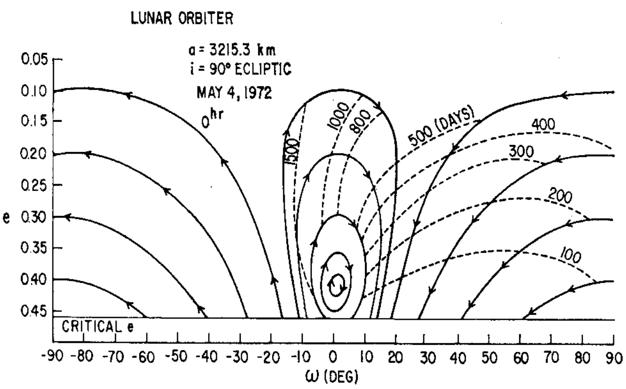

Fig. 3 Eccentricity vs argument of perigee-lunar orbiter.

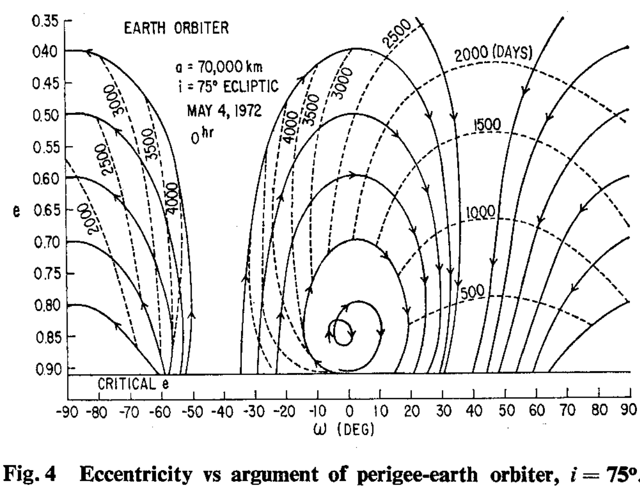

Fig. 4 Eccentricity vs argument of perigee-earth orbiter, $i = 75°$.

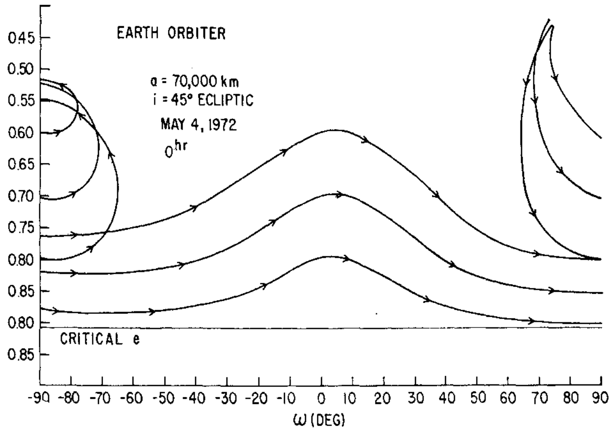

Fig. 5 Eccentricity vs argument of perigee-earth orbiter, $i = 45°$.

initial inclination to the ecliptic. For Earth orbiters, the inclination does not remain as constant as for lunar orbiters, however, the change in inclination is only about 12°; therefore approximate initial conditions may still be obtained but must be finally checked by numerical integration. Total CPU time for Fig. 4 was about 25 min.

Figure 5 is again an Earth orbiter inclined at 45° to the ecliptic and shows stable orbits as far as lifetime is concerned. The intersections of the curves near $\omega = \pm 90°$ are indicative of the fact that inclination does not remain constant, varying as much as 15° for some of the curves in this figure. Again, however, first approximations to initial conditions may still be obtained. Approximately 20 min of CPU time was required for Fig. 5.

Analytical Results

The results obtained thus far have been arrived at by numerically integrating the medium and long-period equations of motion. For a lunar orbiter (and some Earth orbiters) the medium-period terms may be averaged out leaving the long-period equations of motion. For a low lunar orbiter these equations may be integrated analytically away from resonance by the method of successive approximations as in Ref. 14. In the absence of oblateness the solution may be expressed in terms of elliptic integrals (Ref. 1). However, in the case of the initially near circular orbit, the solution to the latter problem may be expressed exactly in terms of the elementary functions. J_2 is considered by introducing a power series expansion in terms of the quantity $[1 - (1 - e^2)^{1/2}]$ which converges quickly for most values of e.

The long period Hamiltonian for a moderately high lunar orbiter can be written as

$$F = \frac{\mu_M}{2L^2} + \frac{1}{4} J_2 \frac{\mu_M{}^4 R_M{}^2}{L^3 G^3}\left(1 - 3\frac{H^2}{G^2}\right) + \frac{n_e{}^2 a^2}{16}\left[\left(5 - 3\frac{G^2}{L^2}\right)\left(3\frac{H^2}{G^2} - 1\right) + 15\left(1 - \frac{G^2}{L^2}\right)\left(1 - \frac{H^2}{G^2}\right)\cos 2g\right] \tag{26}$$

where L, G, H, and g are the usual Delaunay variables.

For a Hamiltonian system

$$\dot{G} = \frac{\partial F}{\partial g}; \qquad \dot{g} = -\frac{\partial F}{\partial G} \tag{27}$$

$$\dot{G} = -\frac{15}{8} n_e{}^2 a^2 \left(1 - \frac{G^2}{L^2}\right)\left(1 - \frac{H^2}{G^2}\right)\sin 2g \tag{28}$$

A relationship between $\sin 2g$ and L, G, H must now be determined. This is accomplished by setting

$$F(L, G, H, g)|_{G=L} = F(L, G, H, g) \tag{29}$$

and solving for $\cos 2g$

$$\cos 2g = \frac{1}{5}\frac{\left(1 - 5\frac{H^2}{G^2}\right)}{\left(1 - \frac{H^2}{G^2}\right)} + \frac{K_0}{1 - \frac{H^2}{G^2}} \times \left[\frac{1 - 3\frac{H^2}{L^2} - \frac{L^3}{G^3}\left(1 - 3\frac{H^2}{G^2}\right)}{\left(1 - \frac{G^2}{L^2}\right)}\right] \tag{30}$$

where

$$K_0 = \frac{4}{15}\left[\frac{\frac{\mu_M}{a} J_2 \left(\frac{R_M}{a}\right)^2}{n_e{}^2 a^2}\right]$$

K_0 is the ratio between the lunar oblateness effect and the terrestrial gravity effect.

Now let

$$f(G) = \frac{1}{G^3}\left(1 - 3\frac{H^2}{G^2}\right) \tag{31}$$

$$F(L) = \frac{1}{L^3}\left(1 - 3\frac{H^2}{L^2}\right) \tag{32}$$

and expand the expression for $\cos 2g$ up to the second power of $(1 - G/L) = X$

$$\cos 2g = \frac{1}{5\left(1 - \frac{H^2}{G^2}\right)}\left[1 - 5\frac{H^2}{L^2}(1 + 2X + 3X^2 + \ldots)\right] + \frac{1}{2}\frac{K_0 L^3}{\left(1 - \frac{H^2}{G^2}\right)}[K_1 + K_2 X + K_3 X^2] \tag{33}$$

where

$$K_1 = \frac{15}{2}\frac{H^2}{L^2} - \frac{1}{2}$$

$$K_2 = \frac{105}{4}\frac{H^2}{L^2} - \frac{5}{4}$$

$$K_3 = \frac{525}{8}\frac{H^2}{L^2} - \frac{55}{24}$$

$$\cos 2g = \frac{A_0 + A_1 X + A_2 X^2}{\left(1 - \frac{H^2}{G^2}\right)} \tag{34}$$

where

$$A_0 = \frac{1}{5} - \frac{H^2}{L^2} + \frac{1}{2} K_0 L^3 K_1$$

$$A_1 = -2\frac{H^2}{L^2} + \frac{1}{2} K_0 K_2 L^3$$

$$A_2 = -3\frac{H^2}{L^2} + \frac{1}{2} K_0 K_3 L^3$$

$$\dot{G} = -\frac{15}{8} n_e{}^2 a^2 \left(1 - \frac{G^2}{L^2}\right)\{B_0 + B_1 X + B_2 X^2\}^{1/2} \tag{35}$$

where

$$B_0 = 1 - \frac{H^2}{L^2} - A_0{}^2$$

$$B_1 = 2A_0A_1 - 4\frac{H^2}{L^2}\left(1 - \frac{H^2}{L^2}\right)$$

$$B_2 = 4\frac{H^2}{L^4} - 6\frac{H^2}{L^2}\left(1 - \frac{H^2}{L^2}\right) + 2A_0A_2 - A_1{}^2$$

In terms of X the equation for $\dot{G}$ becomes

$$\dot{X} = \frac{15}{16}\frac{n_e{}^2}{n^2} X\{C_0 + C_1X + C_2X^2\}^{1/2} \quad (36)$$

where

$$C_0 = B_0$$
$$C_1 = B_1 - B_0$$
$$C_2 = B_0/4 - B_1 + B_2$$

Equation (36) may be integrated

$$\frac{1}{C_0^{1/2}} \log \left[\frac{(C_0 + C_1X + C_2X^2)^{1/2} + C_0^{1/2}}{X} + \frac{C_1}{2C_0}\right]\Bigg|_{X_0}^{X} = \frac{15}{16}\frac{n_e{}^2}{n^2} n(t - t_0) \quad (37)$$

Equation (37) give a time history for the evolution of the canonical pair (G, g) for an initially near circular lunar orbit. The solution is valid for any value for the semimajor axis as long as the following condition is satisfied

$$0 < \left|\left(\frac{1}{5} - \frac{3}{2}K_0\right)\left(\frac{1 - 5\frac{H^2}{L^2}}{1 - \frac{H^2}{L^2}}\right)\right| < 1 \quad (38)$$

These results can be summarized in Table 1. The lifetimes for six different lunar orbiters are computed using Eq. (37). As a check, these same cases are computed using the program POPLAR which is described in a previous section. As can be seen from Table 1, the agreement is excellent considering the simplicity of the analysis.

Summary

The inclusion of medium-period terms in the equations of motion, while not being new in the theory, has not until the present, been available in a rapid program useful for the Earth and moon because of the excessive algebra involved in carrying out the expansions in the parallax factor. The development of an algebraic manipulation routine has not only made this expansion possible, but has allowed the further inclusion of the motion of the disturbing body into the Lagrangian equations. Usually the disturbing body has been held fixed during the averaging process, but for high orbiters this assumption is no longer valid. The algebraic program carries out, automatically, the expansion of the disturbing function, differentiation of the expansion, averaging the equations of motion, and then punches out the resulting equations on computer cards in a Fortran compatible mode. These cards are then inserted directly into the program.

Gravitational harmonics for the Earth and moon are also included by averaging the variations in the elements by means of Gaussian quadratures over one orbit of the satellite. These averaged variations are then included in the total variation of the elements which are numerically integrated to yield a time history of the orbit.

Drag effects, while not used in the present examples for the Earth, may be included in the perturbation model. At present this is limited to a nonrotating atmosphere with no lift present and with the drag force acting as a negative tangential component. The atmospheric density ρ is calculated as a function of altitude.

The program has proven to be a very fast and accurate one. In its high-speed mode of calculating third body perturbations and oblateness (J_2, J_3, J_4) only it has reached speeds of greater than 500 to 1 over numerical integration. When the full gravitational harmonics are included, speeds of 25 to 50 to 1 are still attainable. Such speeds and accuracy make the program extremely useful in parametric studies and in gaining insights into the behavior of planetary orbiters.

In special cases it has been shown that the long-period equations of motions may be solved analytically. The results yield lifetimes of unstable orbits accurate to about 10%.

Table 1 Lifetime comparisons

a_0	e_0	i_0	ω_0	Ω_0	Lifetime [Eq. (37)]	Lifetime (POPLAR)
5214km	0.1	90°	40°	0°	0.92 years	0.99 years
5214	0.1	75°	40°	0°	0.98	1.05
5214	0.2	75°	40°	0°	0.64	0.68
6952	0.1	90°	40°	0°	0.65	0.70
6952	0.1	75°	40°	0°	0.70	0.79
6952	0.2	75°	40°	0°	4.08	0.55

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